

ELEMENTARY PROOFS FOR KATO SMOOTHING ESTIMATES OF SCHRÖDINGER-LIKE DISPERSIVE EQUATIONS

XUWEN CHEN

ABSTRACT. In this expository note, we consider the dispersive equation:

$$i\phi_t = (-\Delta)^{\frac{\beta}{2}}\phi \text{ in } \mathbb{R}^{n+1}, \quad \phi(x, 0) = f(x) \in L^2(\mathbb{R}^n).$$

We prove some extensions and refinements of classical Kato type estimates with elementary techniques.

In this short note, we give easier and unified proofs for certain smoothing estimates of the dispersive equation:

$$i\phi_t = (-\Delta)^{\frac{\beta}{2}}\phi \text{ in } \mathbb{R}^{n+1}, \quad \phi(x, 0) = \phi_0(x) \in L^2(\mathbb{R}^n). \quad (0.1)$$

Theorems 1 and 2 extend the classical Kato estimate:

$$\int_{-\infty}^{\infty} \int_{\mathbb{R}^n} \frac{||\nabla|^\alpha \phi(x, t)|^2}{|x|^{2-2\alpha}} dx dt \leq C \|\phi(\cdot, 0)\|_2^2, \text{ for } \alpha \in [0, \frac{1}{2}) \text{ and } n \geq 3 \quad (0.2)$$

in Kato and Yajima [4], and Ben-Artzi and Klainerman [1] for the free Schrödinger equation ($\beta = 2$ in equation 0.1) and show that estimate 0.2 is in fact an identity whenever the initial data is radial. In particular, this also implies that when $n = 3$ and $\alpha = 0$, the best constant in estimate 0.2 is attained for every $L^2(\mathbb{R}^3)$ radial data (via Simon [6]). As pointed out in Vilela [9], the free Schrödinger endpoint Strichartz estimate for radial data in the case when $n \geq 3$ follows from estimate 0.2. Moreover, the proof of theorem 2 in fact gives theorem 3 which is stated below.

0.1. Statement of the theorems.

Theorem 1. *Let ϕ be the solution to equation 0.1, then for $1 < \beta - 2\alpha < n$, we have*

$$\int_{-\infty}^{\infty} \int_{\mathbb{R}^n} \frac{||\nabla|^\alpha \phi(x, t)|^2}{|x|^{\beta-2\alpha}} dx dt \leq C_{n,\alpha,\beta} \|\phi_0\|_2^2,$$

Moreover, if ϕ_0 is spherically symmetric, then equality holds i.e.

$$\int_{-\infty}^{\infty} \int_{\mathbb{R}^n} \frac{||\nabla|^\alpha \phi(x, t)|^2}{|x|^{\beta-2\alpha}} dx dt = C_{n,\alpha,\beta} \|\phi_0\|_2^2.$$

Remark 1. *As mentioned before, the above estimate when $\beta = 2$, was proved by Kato and Yajima [4] in 1989, Ben-Artzi and Klainerman [1] in 1992. The case $\beta = 2$, $\alpha = 0$ was also mentioned by Herbst [3] and Simon [6] in 1991. Vilela reproved estimate 0.2 to give the endpoint Strichartz estimate for radial data in the case when $n \geq 3$ in [9] in 2001. However, they did not show the equality for radial data. In addition, we will avoid the use of trace lemmas.*

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When $n = 1$, we have the same theorem back, but we have to assume odd initial data.

Theorem 2. *Let ϕ be the solution to equation 0.1 in \mathbb{R}^{1+1} with odd initial data, i.e.*

$$\phi_0(-x) = -\phi_0(x),$$

then for $1 < \beta - 2\alpha \leq 2$, we have the identity

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{||\nabla|^{\alpha} \phi(x, t)|^2}{|x|^{\beta-2\alpha}} dx dt = \frac{2^{\beta-2\alpha} \Gamma(2 - \beta + 2\alpha) \sin\left(\frac{2-\beta+2\alpha}{2}\pi\right)}{\beta(\beta-1-2\alpha)} \|\phi_0\|_2^2.$$

In particular, when $\alpha = 0$, $\beta = 2$, we have

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{|\phi(x, t)|^2}{|x|^2} dx dt = \pi \|\phi_0\|_2^2.$$

Or equivalently, say $\psi(|x|, t)$ solves equation 0.1 when $\beta = 2$ in \mathbb{R}^{3+1} as a 3d radial function, then we have the identity

$$\int_{-\infty}^{\infty} \int_{\mathbb{R}^3} \frac{|\psi(|x|, t)|^2}{|x|^2} dx dt = \pi \|\psi(|\cdot|, 0)\|_{L^2(\mathbb{R}^3)}^2.$$

Remark 2. *Simon showed that the best constant in the classical Kato estimate 0.2 is $\frac{\pi}{n-2}$ when $\alpha = 0$ in [6], but he did not give an explicit ϕ_0 to reach that bound.*

Remark 3. *It is true that if*

$$iu_t = -\Delta u + |x|^2 u \text{ in } \mathbb{R}^{n+1}, \quad (0.3)$$

then

$$\int_0^{2\pi} \int_{\mathbb{R}^n} \frac{|u(x, t)|^2}{|x|^2} dx dt \leq C \|u(\cdot, 0)\|_2^2$$

when $n \geq 3$. Also there is a theorem similar to theorem 2 for equation 0.3 in \mathbb{R}^{1+1} . However, the proof is quite different from what we are dealing with here. See Chen [2].

Remark 4. *Vega and Visciglia also proved a family of identities involving the local smoothing effect for the Schrödinger equation. See Vega and Visciglia [10].*

For $\alpha = \frac{\beta-1}{2}$ and $n = 1$, the proof of theorem 2 in fact reproduces the following result which was part of theorem 4.1 in Kenig, Ponce and Vega [5].

Theorem 3. *Without assuming odd initial data, if $\phi(x, t)$ solves equation 0.1 in \mathbb{R}^{1+1} , then we have*

$$\sup_{x \in \mathbb{R}} \int_{-\infty}^{\infty} \left| |\nabla|^{\frac{\beta-1}{2}} \phi(x, t) \right|^2 dt \leq C \|\phi_0\|_{L^2(\mathbb{R})}^2, \quad \beta > -1.$$

Remark 5. *The above estimate answers exercise 2.56 in Tao [8].*

0.2. **Proof of theorem 1.** It is well known that

$$|\nabla|^\alpha \phi(x, t) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} |\xi|^\alpha e^{ix \cdot \xi} e^{-i|\xi|^\beta t} \hat{\phi}_0(\xi) d\xi,$$

if we choose

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} e^{-ix \cdot \xi} f(x) dx,$$

which gives

$$\|\hat{f}\|_2^2 = (2\pi)^n \|f\|_2^2 \text{ and } \int_{\mathbb{R}^n} e^{-ix \cdot \xi} dx = (2\pi)^n \delta(\xi).$$

Hence we have

$$\begin{aligned} & \int_{-\infty}^{\infty} ||\nabla|^\alpha \phi(x, t)|^2 dt \\ &= \frac{1}{(2\pi)^{2n}} \int_{-\infty}^{\infty} dt \int_{\mathbb{R}^n} d\xi_1 \int_{\mathbb{R}^n} d\xi_2 \left(e^{ix \cdot \xi_1} e^{-i|\xi_1|^\beta t} e^{-ix \cdot \xi_2} e^{i|\xi_2|^\beta t} |\xi_1|^\alpha |\xi_2|^\alpha \hat{\phi}_0(\xi_1) \overline{\hat{\phi}_0(\xi_2)} \right) \\ &= \frac{1}{(2\pi)^{2n}} \int_{\mathbb{S}^{n-1}} dS_{\omega_1} \int_{\mathbb{S}^{n-1}} dS_{\omega_2} \int_0^\infty r_1^{n-1} dr_1 \\ & \quad \lim_{\epsilon \rightarrow 0} \int_0^\infty r_2^{n-1} dr_2 \int_{-\infty}^{\infty} dt \left(e^{ix \cdot (r_1 \omega_1 - r_2 \omega_2)} \eta(\epsilon t) e^{-i(r_1^\beta - r_2^\beta)t} |r_1|^\alpha |r_2|^\alpha \hat{\phi}_0(r_1 \omega_1) \overline{\hat{\phi}_0(r_2 \omega_2)} \right) \\ &= \frac{1}{(2\pi)^{2n}} \int_{\mathbb{S}^{n-1}} dS_{\omega_1} \int_{\mathbb{S}^{n-1}} dS_{\omega_2} \int_0^\infty r_1^{n-1} dr_1 \\ & \quad \lim_{\epsilon \rightarrow 0} \int_0^\infty r_2^{n-1} dr_2 \left(e^{ix \cdot (r_1 \omega_1 - r_2 \omega_2)} \hat{\eta}_\epsilon(r_1^\beta - r_2^\beta) |r_1|^\alpha |r_2|^\alpha \hat{\phi}_0(r_1 \omega_1) \overline{\hat{\phi}_0(r_2 \omega_2)} \right) \\ &= \frac{1}{(2\pi)^{2n}} \int_{\mathbb{S}^{n-1}} dS_{\omega_1} \int_{\mathbb{S}^{n-1}} dS_{\omega_2} \int_0^\infty r_1^{n-1} dr_1 \\ & \quad \lim_{\epsilon \rightarrow 0} \int_0^\infty v^{\frac{n-1}{\beta}} v^{\frac{1}{\beta}-1} \frac{dv}{\beta} \left(e^{ix \cdot (r_1 \omega_1 - v^{\frac{1}{\beta}} \omega_2)} \hat{\eta}_\epsilon(r_1^\beta - v) |r_1|^\alpha v^{\frac{\alpha}{\beta}} \hat{\phi}_0(r_1 \omega_1) \overline{\hat{\phi}_0(v^{\frac{1}{\beta}} \omega_2)} \right) \\ &= \frac{1}{\beta} \frac{1}{(2\pi)^n} \int_{\mathbb{S}^{n-1}} dS_{\omega_1} \int_{\mathbb{S}^{n-1}} dS_{\omega_2} \int_0^\infty \left(r_1^{n-\beta+2\alpha} e^{ix \cdot (r_1 \omega_1 - r_1 \omega_2)} \hat{\phi}_0(r_1 \omega_1) \overline{\hat{\phi}_0(r_1 \omega_2)} \right) r_1^{n-1} dr_1 \end{aligned} \tag{0.4}$$

where η is a suitable bump function i.e. $\hat{\eta}_\epsilon(\xi) = \frac{1}{\epsilon} \hat{\eta}\left(\frac{\xi}{\epsilon}\right)$ is an approximation to $(2\pi)^n \delta(\xi)$. This approximation of identity is used in order to avoid $\delta(r_1^\beta - r_2^\beta)$ in some dimensions.

Whence

$$\begin{aligned} & \int_{-\infty}^{\infty} \int_{\mathbb{R}^n} \frac{||\nabla|^\alpha \phi(x, t)|^2}{|x|^{\beta-2\alpha}} dx dt \\ &= \frac{1}{\beta} \frac{1}{(2\pi)^n} \int_{\mathbb{S}^{n-1}} dS_{\omega_1} \int_{\mathbb{S}^{n-1}} dS_{\omega_2} \int_{\mathbb{R}^n} \frac{e^{ix \cdot r_1(\omega_1 - \omega_2)}}{|x|^{\beta-2\alpha}} dx \\ & \quad \int_0^\infty \left(r_1^{n-\beta+2\alpha} \hat{\phi}_0(r_1 \omega_1) \overline{\hat{\phi}_0(r_1 \omega_2)} \right) r_1^{n-1} dr_1 \\ &= c_{n,\alpha} \int_0^\infty r_1^{n-1} dr_1 \int_{\mathbb{S}^{n-1}} dS_{\omega_1} \int_{\mathbb{S}^{n-1}} dS_{\omega_2} \frac{1}{|\omega_1 - \omega_2|^{n-\beta+2\alpha}} \hat{\phi}_0(r_1 \omega_1) \overline{\hat{\phi}_0(r_1 \omega_2)}, \end{aligned}$$

excluding the case when $\beta - 2\alpha = n$ due to the fact that $|x|^{-n}$ is not a tempered distribution in n d.

Because $n - \beta + 2\alpha < n - 1$ if $1 < \beta - 2\alpha$, the above computation concludes the proof of theorem 1.

Remark 6. *The steps in the above proof can be traced back to Sjölin [7] in which the author proved various other local smoothing estimates for the free Schrödinger equation. In the case we are dealing with here, the computation is carried out explicitly.*

0.3. Proof of theorems 2 and 3. Relation 0.4 reads

$$\begin{aligned}
& \int_{-\infty}^{\infty} ||\nabla|^\alpha \phi(x, t)|^2 dt \\
&= \frac{1}{4\pi^2} \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} d\xi_1 \int_{-\infty}^{\infty} d\xi_2 \left(e^{ix\xi_1} e^{-i|\xi_1|^\beta t} e^{-ix\xi_2} e^{i|\xi_2|^\beta t} |\xi_1|^\alpha |\xi_2|^\alpha \hat{\phi}_0(\xi_1) \overline{\hat{\phi}_0(\xi_2)} \right) \\
&= \frac{1}{4\pi^2} \int_{-\infty}^{\infty} d\xi_1 \lim_{\epsilon \rightarrow 0} \int_{-\infty}^{\infty} d\xi_2 \left(e^{ix(\xi_1 - \xi_2)} \hat{\eta}_\epsilon(|\xi_1|^\beta - |\xi_2|^\beta) |\xi_1|^\alpha |\xi_2|^\alpha \hat{\phi}_0(\xi_1) \overline{\hat{\phi}_0(\xi_2)} \right) \\
&= \frac{1}{4\pi^2} \int_0^\infty d\xi_1 \lim_{\epsilon \rightarrow 0} \int_0^\infty d\xi_2 + \frac{1}{4\pi^2} \int_{-\infty}^0 d\xi_1 \lim_{\epsilon \rightarrow 0} \int_{-\infty}^0 d\xi_2 \\
&\quad + \frac{1}{4\pi^2} \int_0^\infty d\xi_1 \lim_{\epsilon \rightarrow 0} \int_{-\infty}^0 d\xi_2 + \frac{1}{4\pi^2} \int_{-\infty}^0 d\xi_1 \lim_{\epsilon \rightarrow 0} \int_0^\infty d\xi_2
\end{aligned}$$

With the same procedure in the proof of theorem 1, we deduce

$$\begin{aligned}
& \int_{-\infty}^{\infty} \frac{1}{|x|^{2-2\alpha}} dx \int_{-\infty}^{\infty} ||\nabla|^\alpha \phi(x, t)|^2 dt \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{|x|^{\beta-2\alpha}} dx \left(\int_0^\infty \frac{1}{\beta |\xi_1|^{\beta-1-2\alpha}} \hat{\phi}_0(\xi_1) \overline{\hat{\phi}_0(\xi_1)} d\xi_1 + \int_{-\infty}^0 \frac{1}{\beta |\xi_1|^{\beta-1-2\alpha}} \hat{\phi}_0(\xi_1) \overline{\hat{\phi}_0(\xi_1)} d\xi_1 \right. \\
&\quad \left. + \int_0^\infty \frac{e^{2ix\xi_1}}{\beta |\xi_1|^{\beta-1-2\alpha}} \hat{\phi}_0(\xi_1) \overline{\hat{\phi}_0(-\xi_1)} d\xi_1 + \int_{-\infty}^0 \frac{e^{2ix\xi_1}}{\beta |\xi_1|^{\beta-1-2\alpha}} \hat{\phi}_0(\xi_1) \overline{\hat{\phi}_0(-\xi_1)} d\xi_1 \right) \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{|x|^{\beta-2\alpha}} \int_{-\infty}^{\infty} \frac{1 - e^{2ix\xi_1}}{\beta |\xi_1|^{\beta-1-2\alpha}} \left| \hat{\phi}_0(\xi_1) \right|^2 d\xi_1 \\
&= \frac{1}{2\beta\pi} \int_{-\infty}^{\infty} d\xi_1 \frac{\left| \hat{\phi}_0(\xi_1) \right|^2}{|\xi_1|^{\beta-1-2\alpha}} \int_{-\infty}^{\infty} dx \frac{1 - \cos 2x\xi_1}{|x|^{\beta-2\alpha}}
\end{aligned}$$

because $\hat{\phi}_0$ is odd if ϕ_0 is odd. However,

$$\begin{aligned}
& \int_{-\infty}^{\infty} \frac{1 - \cos 2x\xi_1}{|x|^{\beta-2\alpha}} dx \\
&= 2 \int_0^\infty \frac{1 - \cos 2x\xi_1}{x^{\beta-2\alpha}} dx \\
&= \frac{2 \cdot 2\xi_1}{\beta - 1 - 2\alpha} \int_0^\infty \frac{\sin 2x\xi_1}{x^{\beta-1-2\alpha}} dx \\
&= \frac{2 \cdot 2 |\xi_1|}{\beta - 1 - 2\alpha} \frac{\Gamma(2 - \beta + 2\alpha) \sin\left(\frac{2 - \beta + 2\alpha}{2}\right) \pi}{(2 |\xi_1|)^{2 - \beta + 2\alpha}} \\
&= \frac{2^{\beta-2\alpha} \Gamma(2 - \beta + 2\alpha) \sin\left(\frac{2 - \beta + 2\alpha}{2}\right) \pi}{\beta - 1 - 2\alpha} |\xi_1|^{\beta-1-2\alpha}
\end{aligned}$$

valid when $1 < \beta - 2\alpha \leq 2$ i.e.

$$\begin{aligned} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{||\nabla|^{\alpha} \phi(x, t)|^2}{|x|^{\beta-2\alpha}} dx dt &= \frac{2^{\beta-2\alpha} \Gamma(2 - \beta + 2\alpha) \sin\left(\frac{2-\beta+2\alpha}{2}\pi\right)}{2\beta(\beta-1-2\alpha)\pi} \|\hat{\phi}_0\|_2^2 \\ &= \frac{2^{\beta-2\alpha} \Gamma(2 - \beta + 2\alpha) \sin\left(\frac{2-\beta+2\alpha}{2}\pi\right)}{\beta(\beta-1-2\alpha)} \|\phi_0\|_2^2. \end{aligned}$$

So theorem 2 is concluded. Notice that relation 0.4 becomes

$$\int_{-\infty}^{\infty} ||\nabla|^{\alpha} \phi(x, t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1 + e^{2ix\xi_1}}{\beta |\xi_1|^{\beta-1-2\alpha}} \left| \hat{\phi}_0(\xi_1) \right|^2 d\xi_1$$

if the initial data ϕ_0 is even. Via the odd-even decomposition, we have also proven theorem 3.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MARYLAND, COLLEGE PARK, MD 20742
E-mail address: chenxuwen@math.umd.edu